THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT2230A Complex Variables with Applications 2017-2018
Suggested Solution to Assignment 8

§57) 1) b) By Cauchy Integral Formula,

$$\int_C \frac{\cos z}{z(z^2+8)}dz = \int_C \frac{\cos z/(z^2+8)}{z}dz = 2\pi i\left(\frac{\cos 0}{0^2+8}\right) = \frac{\pi i}{4}$$

e) By Cauchy Integral Formula,

$$\int_C \frac{\tan z/2}{(z-x_0)^2}dz = 2\pi i\left[\frac{d}{dz}\tan(z/2)\right]_{z=x_0} = \pi i(\sec^2(x_0/2))$$

§57) 2) b) By Cauchy Integral Formula,

$$\int_C \frac{1}{(z^2+4)^2}dz = \int_C \frac{1/(z+2i)^2}{(z-2i)^2}dz = 2\pi i\left[\frac{d}{dz}\frac{1}{(z+2i)^2}\right]_{z=2i} = 2\pi i\left[\frac{-2}{(z+2i)^3}\right]_{z=2i} = \frac{\pi}{16}$$

§57) 4) If z is inside C, by Cauchy Integral Formula,

$$g(z) = \int_C \frac{s^3+2s}{(s-z)^3}ds = \frac{2\pi i}{2!}\left[\frac{d^2}{ds^2}(s^3+2s)\right]_{s=z} = 6\pi iz$$

If z is outside C, then the integrand is analytic on and inside C. By Cauchy-Goursat Theorem, we have $g(z)=0$.

§57) 5) Case 1: Assume that z_0 is inside C.

By Cauchy Integral Formula, we have

$$\int_C \frac{f'(z)}{z-z_0}dz = 2\pi i f'(z_0)$$

Similarly, we also have

$$\int_C \frac{f(z)}{(z-z_0)^2}dz = 2\pi i\left[\frac{d}{dz}f(z)\right]_{z=z_0} = 2\pi i f'(z_0)$$

Altogether we have

$$\int_C \frac{f'(z)}{z-z_0}dz = \int_C \frac{f(z)}{(z-z_0)^2}dz$$

Case 2: Assume that z_0 is outside C.

Since the integrands on both sides are analytic on and inside C, by Cauchy-Goursat Theorem, both of them are zeros. In particular they are equal.

§57) 7) For any real number a, by Cauchy Integral Formula, we have

$$\int_C \frac{e^{az}}{z}dz = 2\pi i e^{a(0)} = 2\pi i$$

On the other hand, we also have

$$\begin{aligned}\int_C \frac{e^{az}}{z}dz &= \int_{-\pi}^{\pi} \frac{e^{a\cos\theta+ia\sin\theta}}{e^{i\theta}} ie^{i\theta}d\theta \\ &= \int_{-\pi}^{\pi} e^{a\cos\theta}[\cos(a\sin\theta)+i\sin(a\sin\theta)]id\theta \\ &= -\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)d\theta + i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta\end{aligned}$$

As a result, by comparing the imaginary part, we have

$$\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = 2\pi$$

Finally, since the integrand is an even function, we have

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = \pi$$

§59) 1) Consider the function $g(z) = \exp[f(z)]$. Since $f(z)$ is entire, $g(z)$ is also entire. Furthermore, $|g(z)| = |\exp[f(z)]| = \exp(\operatorname{Re} f(z)) \leq \exp(u_0)$ for any $z \in \mathbb{C}$. Therefore, by Liouville's Theorem, $g(z)$ must be a constant function, i.e. $g(z) = C$ for some constant $C \in \mathbb{C}$. In particular, we have $f(z) \in \log C = \ln|C| + i\arg C$. Then by continuity, $f(z)$ must be a constant function.

§59) 4) Note that $|f(z)|^2 = \sin^2 x + \sinh^2 y$. For $x \in [0, \frac{\pi}{2}]$, the maximum of $\sin^2 x$ is $\sin^2(\frac{\pi}{2}) = 1$. On the other hand, for $y \in [0,1]$, since $\sinh^2 y$ is a strictly increasing function in y, the maximum of $\sinh^2 y$ is $\sinh^2 1$. Overall, the maximum of the modulus of the function $f(z)$ in R attains at the point $\frac{\pi}{2} + i$.

§61) 3) Given that $\lim_{n\to\infty} z_n = z$. Then for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|z_n - z| < \epsilon$ for any $n \geq N$. In particular, for any $n \geq N$, we have $||z_n| - z| \leq |z_n - z| < \epsilon$. This proves that $\lim_{n\to\infty} |z_n| = |z_0|$.

§61) 4) Since $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ whenever $|z| < 1$, put $z = re^{i\theta}$ with $0 < r < 1$. Then we have

$$\begin{aligned}\sum_{n=0}^{\infty} r^n e^{in\theta} &= \frac{1}{1-re^{i\theta}} \\ 1 + \sum_{n=1}^{\infty} r^n e^{in\theta} &= \frac{1}{1-r\cos\theta - ir\sin\theta} \\ \sum_{n=1}^{\infty} r^n\cos(n\theta) + i\sum_{n=1}^{\infty} r^n\sin(n\theta) &= \frac{1-r\cos\theta+ir\sin\theta}{1-2r\cos\theta+r^2} - 1 \\ \sum_{n=1}^{\infty} r^n\cos(n\theta) + i\sum_{n=1}^{\infty} r^n\sin(n\theta) &= \frac{r\cos\theta - r^2 + ir\sin\theta}{1-2r\cos\theta+r^2}\end{aligned}$$

By comparing the real and imaginary part on both sides, we get

$$\sum_{n=1}^{\infty} r^n\cos(n\theta) = \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2} \text{ and } \sum_{n=1}^{\infty} r^n\sin(n\theta) = \frac{r\sin\theta}{1-2r\cos\theta+r^2}$$

The equations are clearly true when $r = 0$.